Parallel Wire Transmission Lines Characteristic Impedance Formulas

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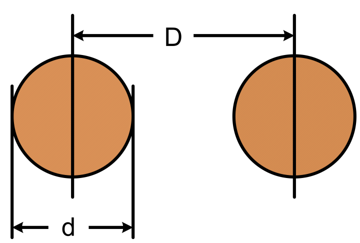
Where it matters, all units are taken to be MKS in this article. Conductors are assumed to have infinite conductivity. *Natural* logarithms are always indicated by the ln() function. Logarithms in any other base use the $\log_b()$ function, and will always have a base, b, specified. Impedances discussed here concern only the lowest order TEM mode of propagation. Predicting the appearance of other modes based on line geometry and frequency of operation is beyond the scope of this article.

Round Conductors



The conductor diameter is represented by the symbol d, and the center-to-center spacing by D. The ratio D/d is unitless, and these dimensions may be expressed in any convenient units (as long as both are expressed in the same unit).

There is a well known, closed form solution to Maxwell's equations for this case. The characteristic impedance when the conductors are embedded in an infinite medium with permeability μ and permittivity ϵ is

$$Z_o = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon}}\cosh^{-1}\frac{D}{d}$$

In air, where relative permittivities and permeabilities are approximately unity,

$$Z_o = \frac{1}{\pi}\sqrt{\frac{\mu_o}{\epsilon_o}}\cosh^{-1}\frac{D}{d} \approx 119.92\cosh^{-1}\frac{D}{d}$$

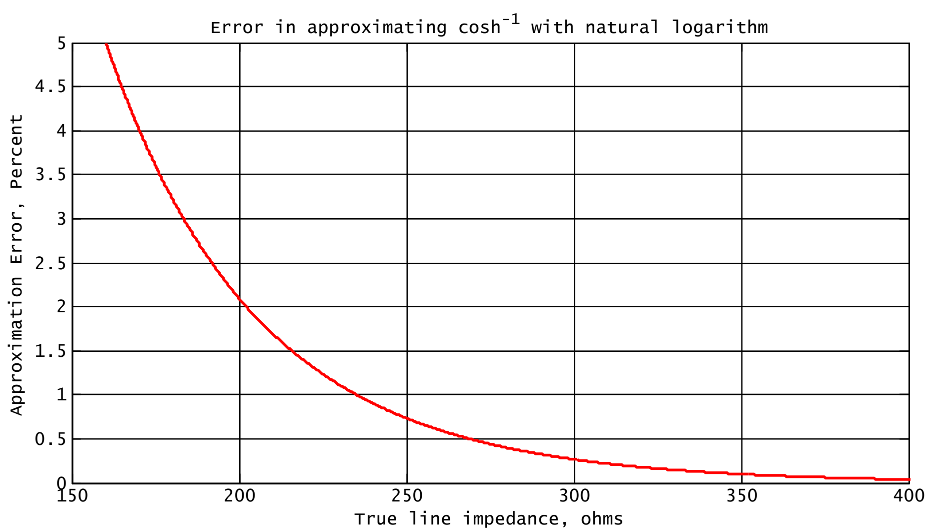
The value of the square root term is the impedance of free space (376.73...Ω), so the multiplier is that impedance divided by π.

A Common Approximation

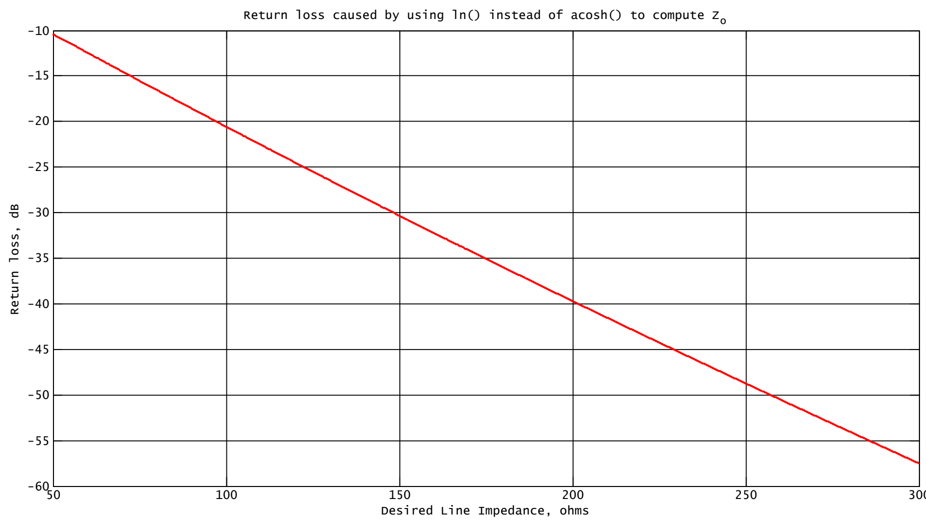
It is not uncommon to find formulas where the hyperbolic arc-cosine has been replaced with the natural logarithm.

$$\cosh^{-1}\frac{D}{d} \approx \ln\frac{D}{d/2}$$

This change works okay for larger values of D/d, but for narrow conductor spacing significant errors are introduced. For unknown reasons, substitution of ln() for $\cosh^{-1}()$ seems to be particularly common in amateur radio circles.

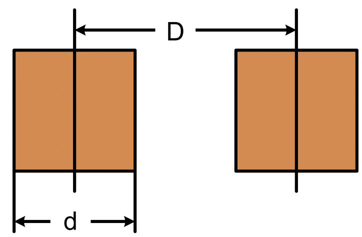


The graph above shows how much error is introduced in the line impedance by using this approximation. The next graph below shows this in terms of return loss due to the impedance error.



For line impedances of 200 ohms or more, the error introduced will result in a return loss no worse than 40dB, and for most purposes this is entirely acceptable. For a line impedance of 100 ohms, the error results in a return loss of about 20dB; this is enough start being a concern in some situations.

Square Conductors



The impedance of a square conductor parallel wire transmission line, with conductors of dimension d on a side, with center-to-center spacing of D might have a closed form solution, but we are not aware of one, nor have we tried to work one out.

For this geometry, E-M simulations have been carefully performed over a range of geometries. The simulated impedance is nearly linear in $(\cosh^{-1} D/d)$. Three equations (one linear and two second-order polynomials) have been fit to the data over different ranges of D/d. Simulations covered a total D/d range from 1.05 to 29, which is an approximate impedance range of 16 to 470 ohms.

1. A linear fit for $D/d > 1.15$ which is useful for most purposes.
2. A more accurate quadratic fit for small values of D/d.
3. A more accurate quadratic fit for larger values of D/d.

The three curve fits are:

$$\Phi = \cosh^{-1} \frac{D}{d}$$

$$Z_{o1}(\Phi) = 121.73\Phi - 25.75 \,|_{D/d>1.15} \tag{1}$$

$$Z_{o2}(\Phi) = 39.82\Phi^2 + 70.56\Phi - 10.23 \,\big|_{1.05\geq D/d\leq 1.25} \tag{2}$$

$$Z_{o3}(\Phi) = -0.878\Phi^2 + 125.60\Phi - 28.86\big|_{D/d\geq 1.25} \tag{3}$$

These may be easily solved for Φ given a desired value of Z_o, and the line spacing determined as $D/d = \cosh\Phi$.

For convenience, the solutions for equations (2) and (3) are provided below. Note that the sign on the square root is not the same in both equations.

$$\Phi_2(Z_o) = -0.88 + \sqrt{1.0418 + 0.0251 Z_o}$$

$$\Phi_3(Z_o) = 71.52 - \sqrt{5083.5862 - 1.1390 Z_o}$$

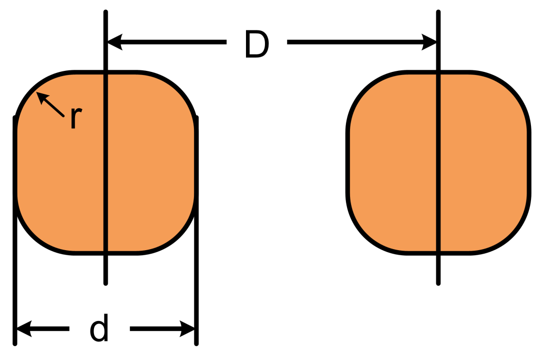
The performance of equations (1-3) in predicting the simulated values is shown in the table below.

Equation	Z_o range	RMS Error	Max Error
1	40 and up	0.8%	1.7%
2	16 to 58	0.3%	0.7%
3	58 and up	0.16%	0.4%

It should be safe to extrapolate equations (1) and (3) for values of $D/d > 29$, but only as long as additional modes of propagation are not excited.

Taking (2) below D/d values of 1.05 may not be accurate. However, this represents a 16Ω line which is not really practical anyway.

Square Conductors with Rounded Corners



Rounding the corners of square conductors with a smooth radius, r, increases Z_o by roughly the same number of ohms, regardless of the actual line impedance. Therefore, it has more relative effect on low impedance geometries.

Rounded corners can be characterized by their radius normalized to the conductor width, r/d. A rounding radius equal to half the line width results in a perfectly round conductor. Based on simulations of a line with D/d=1.6 (roughly 100 ohms), the following curve fit estimates how much impedance is added by rounding the corners, as a function of the normalized corner radius.

$$\Delta Z_o\left(\frac{r}{d}\right) = 53.06\left(\frac{r}{d}\right)^2 + 20.97\,\frac{r}{d} + 0.09\,\bigg|_{0\le r/d\le 0.5}$$

For this geometry ($D/d = 1.6$), here's how the approximation compares to simulated and theoretical values.

Method	Z_o (Square)	Z_o (Round)
Std Formula		125.55
Simulation	101.58	125.37
BFSL	101.69	
Polynomial	101.68	

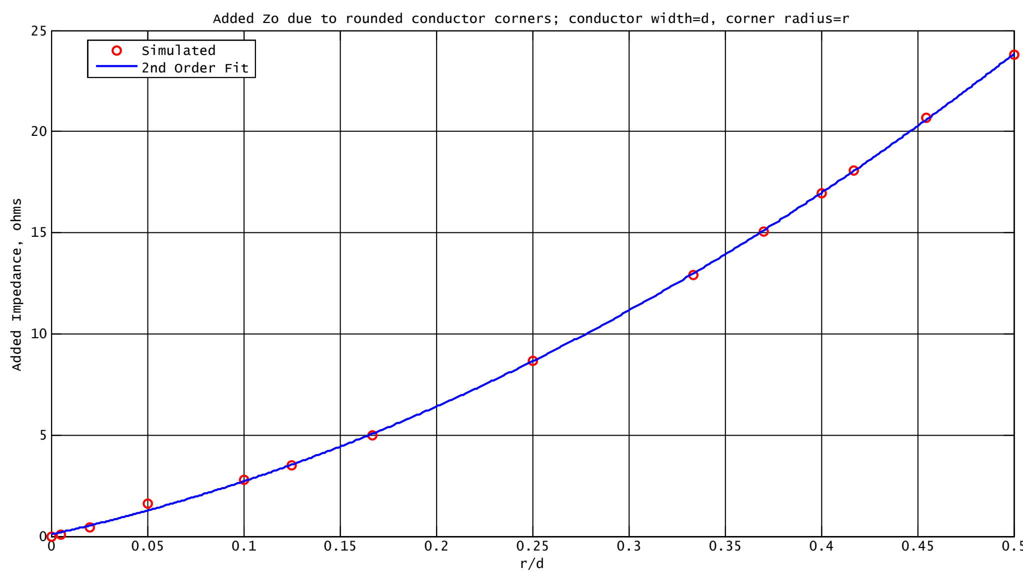
Above, in the first column, simulation data for square conductors is compared to the curve fits. Best-fit straight line (BFSL) and second-order polynomial fits agree well with the simulation data, to within about 0.1 ohm.

In the second column, E-M simulation of round conductors is compared with the closed form solution, and agrees to better than 0.2 ohms.

The increase in impedance in going from square to round conductors, as simulated and as predicted by the second order polynomial fit for ΔZ_o is shown below. These agree to better than 0.2 ohms.

Added Impedance	
Simulated	Predicted from polynomial
23.97	23.83

This graph shows the simulation data for added impedance, ΔZ_o (red dots) and the second order polynomial curve fit.



In summary, to estimate the impedance of a square conductor line with rounded corners, first use one of the curve fits Z_{o1}, Z_{o2}, Z_{o3} to obtain the impedance of the line made with square conductors, then add the impedance predicted by this curve fit:

$$Z_o(D, d, r) = \Delta Z_o \left(\frac{r}{d} \right) + Z_{on} \left(\cosh^{-1} \frac{D}{d} \right) \bigg|_{n=1,2,3}$$

For example, find the impedance of a line made from square conductors, 10mm on a side with 2mm radius corners, spaced 20mm apart.

$$
\begin{aligned}
\Phi &= \cosh^{-1}\frac{20}{10} \approx 1.3170 \\
Z_{o3} &= -0.878\Phi^2 + 125.60\Phi - 28.86 \approx 135.03 \\
\frac{r}{d} &= \frac{2}{10} = 0.2 \\
\Delta Z_o &= 53.06 \times 0.2^2 + 20.97 \times 0.2 + 0.09 \approx 6.41 \\
Z_o &\approx 135.03 + 6.41 \approx 141\Omega
\end{aligned}
$$

To find the line spacing for a desired impedance using square conductors with rounded conductors, given values of r, d, first subtract the rounding correction from the desired impedance, then use one of the solutions to the square conductor curve fits.

$$D = d\ \cosh\left[\Phi_n\left(Z_o - \Delta Z_o\left(\frac{r}{d}\right)\right)\right]_{n=1,2,3}$$

Example: find the line spacing for a 125-ohm line using the rounded conductors from the above example. As calculated above, the rounded corners will increase the impedance by 6.41 ohms, so the line spacing for a 118.59 ohm line is required (125 - 6.41 = 118.59). Solving for Φ, and then the line spacing,

$$
\begin{aligned}
\Phi &= 71.52 - \sqrt{5083.5862 - 1.1390 \times 118.59} \approx 1.1743 \\
\frac{D}{d} &= \cosh 1.1743 \approx 1.7725 \\
D &= 1.7725 \times 10\text{mm} \approx 17.73\text{mm}
\end{aligned}
$$

E-M Simulation Specifics

Simulations were performed using perfect electrical conductors and waveguide ports at each end of the transmission line. The ports were sized to be 60 times the largest dimension of the TL profile. Using larger port areas did not seem to have much effect on the calculated port impedances.

Ports were divided into two concentric squares, with a finer mesh applied to the inner square to improve the accuracy of port impedance calculations.

The solver setup specified that residuals should be kept below -60dB, and simulations were run at 50MHz with lines having $d = 0.1$ inch.

No real-world lines were fabricated for testing to verify the simulations of square lines, although simulations of round lines agreed well with theory.

Modes

As mentioned at the start of this article, it's beyond the scope to say much about propagation modes other than the lowest order TEM mode.

Just for background however, we report on the propagating modes for a square conductor line with $d = 0.1$ inch and $D = 3$ inch. Simulation results indicate there are two such modes, one TEM and one TM with a cutoff of 32MHz. There are also at least two other non-propagating modes (one TE and one TM) with a 40dB distance of about 14 feet.

Simulation Check

As a validation of E-M simulations performed for this study, several transmission lines with round conductors were simulated, covering a range of impedances from 38 to 350 ohms. Since the "right answer" is known in this case, it provides a good check on the simulation setup and accuracy. A BFSL fit to the simulation data yielded this formula for the characteristic impedance, which is very close to theory.

$$Z_o = 119.98 \cosh^{-1}\left(\frac{D}{d}\right) - 0.11$$

This formula fits the simulated impedances with an RMS error of 0.16% and a maximum error of 0.6%.

This document was revised in the fall of 2022, adding text to clarify some tabular data, and two examples of calculations for square conductors with rounded corners.