

Digitally Convenient Chirps

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This article describes sinusoidal, digitally generated analytic¹ chirp signals whose frequency varies linearly with time. This appears in the frequency domain as a set of equally spaced tones, and is useful as a test signal in a multitude of situations.

A single frequency may be generated analytically like this:

$$x(t) = e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

This may be either a single negative or positive frequency. Being analytic, there is no requirement that the signal contain negative and positive frequencies in equal measure. The real and imaginary parts of this analytic signal are applied to the I and Q modulator inputs of a signal generator for up-conversion to a carrier frequency. These may also be sampled and the samples fed to I and Q digital-to-analog converters (DACs) for up-conversion.

A Unique Chirp

The chirp waveform described in this article has the following qualities.

- It sweeps linearly in time from some negative frequency, $-B$ to the positive frequency B , then returns to the start frequency and repeats.
- The phase is equal at the beginning and end of the waveform.
- Both I and Q waveforms are continuous from end to beginning; there are no discontinuities when played repetitively.

¹Analytic signals are complex-valued.

- One of the waveforms will have continuous derivatives from end to beginning; the other will not.
- In the time domain, both I and Q waveforms and the up-converted modulated waveform have a crest factor of one.
- The signal appears in the up-converted frequency domain as N equal-amplitude carriers with a parabolic phase progression between them.

This is a very convenient waveform for generating a picket-fence test signal to measure frequency responses because it provides a set of equal-amplitude test tones with an ideal crest factor.

Mathematical Derivation

Here we derive the analytic baseband waveform. In general, the radian frequency of a sinusoidal, linear chirp with bandwidth $2B$ radians/second is:

$$\omega(t) = -B + rt$$

where r is the chirp rate in radians per second squared. The phase of the signal is obtained by integrating the frequency:

$$\phi(t) = \int \omega(t)dt = -Bt + \frac{rt^2}{2} + C$$

We arbitrarily set the constant of integration (the starting phase) to zero. This signal has a frequency equal to $-B$ at time zero. If we want the chirp signal to have the frequency $+B$ at some later time, T then the chirp rate is uniquely determined:

$$r = \frac{2B}{T}$$

Conversely, given a chirp rate the time required to sweep from $-B$ to $+B$ is determined:

$$T = \frac{2B}{r}$$

The time domain description of this chirp signal is

$$x(t) = e^{j\phi(t)}, \quad \text{where} \quad \phi(t) = -Bt + \frac{rt^2}{2} + C$$

For phase continuity we would like the phase at the end of the chirp to be an integral multiple of $2\pi + C$ (or 2π in this case since $C = 0$). Evaluating the above definition of $\phi(t)$ at the end of the chirp (at time T):

$$\phi(T) = -B\frac{2B}{r} + \frac{r}{2} \left[\frac{2B}{r} \right]^2 = 0$$

Therefore, continuous phase between end and beginning of chirp are guaranteed by design.

This can also be seen intuitively by realizing that since the signal spends an equal amount of time at positive and negative frequencies, all the phase accumulated during the segment from frequency $-B$ to zero is *unwound* during the symmetric second half which ramps from zero frequency to $+B$.

Sampling the Chirp

Sampling this signal at the proper rate yields the desired digital analytic signal, with continuous phase, a crest factor of one and a picket fence of equal amplitude tones in the frequency domain. Note that sampling at the wrong rate will yield a mess – even if it meets the Nyquist criteria.

The Nyquist criteria requires the sampling rate be at least twice the highest frequency content of the signal. The sample rate which exactly meets this criteria is

$$f_s = 2 \frac{B}{2\pi} = \frac{B}{\pi}$$

and the corresponding sampling interval is

$$t_s = \frac{1}{f_s} = \frac{\pi}{B}$$

Finally, the number of sampled points during the chirp is equal to the chirp duration divided by the sampling interval:

$$N = \frac{T}{t_s} = \frac{2B}{r} \frac{B}{\pi} = \frac{2B^2}{\pi r}$$

We wish N to come out as an integer and may choose it to be some convenient value, such as a power of two. This then can be used to specify the chirp rate:

$$r = \frac{2B^2}{\pi N}$$

It is also possible to sample the same chirp at integer multiples of f_s . That's all there is to it.

Example

Figure 1 shows the I and Q waveforms for an example chirp signal. The top graph shows one entire cycle from beginning at frequency $-B$ to end at frequency $+B$. The middle plot zooms in on the center when the chirp passes through zero frequency. In the bottom plot, two chirp waveforms have been concatenated and this zooms in on the end of one waveform (at $+B$)

as it transitions to the next cycle beginning at frequency $-B$. It can be seen here that the I signal is not only continuous across this boundary, but its derivatives are also continuous. The Q signal is also continuous across the boundary but has a discontinuity in the derivatives.

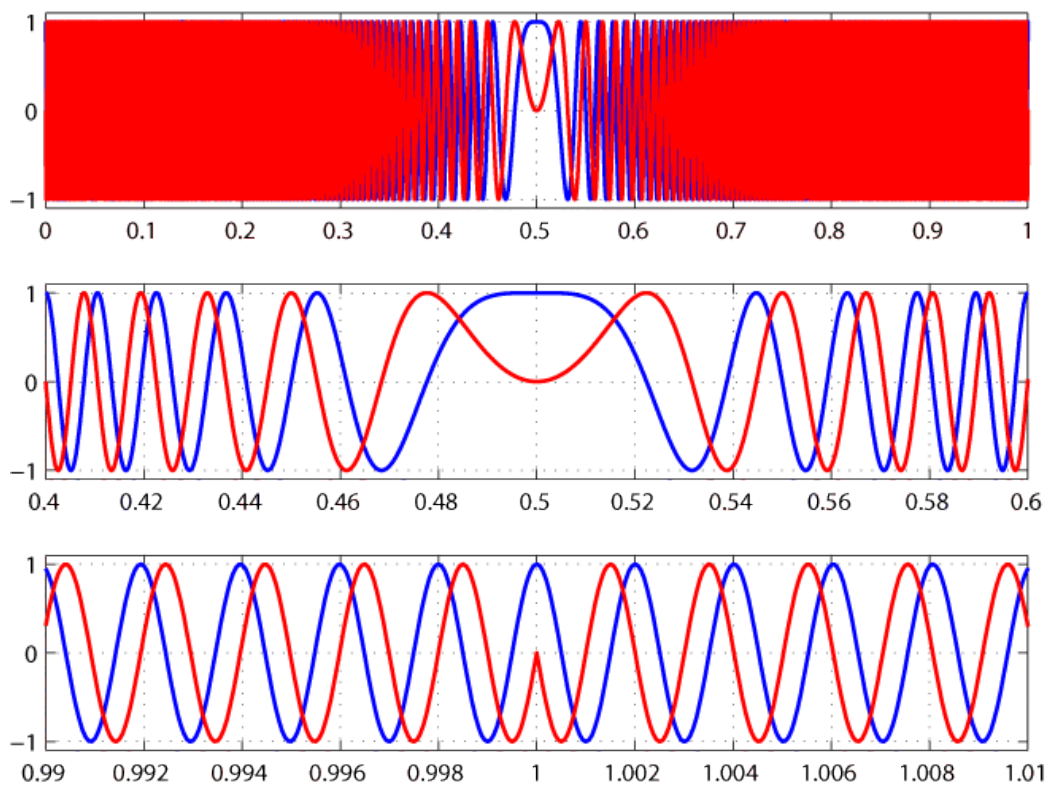


Figure 1: Example Chirp Signal

Summary

To create the sampled chirp signal, only two parameters are required:

- Radian half-bandwidth, B
- DFT length, N , also equal to the number of tones in frequency domain

From this, compute

$$r = \frac{2B^2}{N}$$

$$t_s = \frac{\pi}{B}$$

$$t_k = t_s k \mid k = \{0..N - 1\}$$

$$\phi_k = -Bt_k + rt_k^2$$

$$x_k = e^{j\phi_k}$$

The DFT of the sequence x will have uniform amplitudes in all bins with a parabolic phase progression between bins. Do not forget that bandwidths, frequencies and rates above are *radian* values.

Sample Matlab[®]/Octave Code

This code has been verified to work in Matlab[®], but some changes may be required to make it run on Octave.

```
%
% Chirp parameters
%
f=500;           % chirp half-bandwidth (e.g. in Hz)
N=64;           % time domain sequence length
%
% Chirp generation
%
B=2*pi*f;       % radian half-bandwidth
r=2*B^2/(N*pi); % chirp rate
T=2*B/r;        % total time to go from -B to +B (seconds)
dt=T/N;         % sequence sampling interval (seconds)
fs=1/dt;        % sampling frequency (Hz)
t=(0:N-1)*dt;   % time at sampling points
phi=-B*t+r*t.^2/2; % radian phase versus time
x=exp(1i*phi);   % analytic time-domain chirp
Imod=real(x);   % digital samples for I modulator input
Qmod=imag(x);   % samples for Q input
%
% A little testing/analysis
%
X=fft(x)*2/N;
Xmag=abs(X);     % all magnitudes will be the same
Xarg=unwrap(angle(X)); % phase progression will be parabolic
```

Semtech LoRa[®] Modulation

The analytic chirps described herein are most likely the same signals used by Semtech in their LoRa[®] modulation scheme. Each symbol in an alphabet is one of these chirp signals with a unique starting phase. For example, in a 64-symbol alphabet there will be 64 unique starting phases. The receiver decodes each symbol by estimating its starting phase.

The *spreading factor* as defined by Semtech is the base-2 logarithm of the number of tones in the chirp signal, which is also the number symbols (and thus, unique starting phases) in the alphabet.

At low data rates (high spreading factors), clock drift between transmitter and receiver may result in enough phase error to cause symbol identification errors. Semtech's low data rate optimization option reduces the number of unique starting phases (and thus symbols in the alphabet) by a factor of four. The distance between adjacent symbols is thus increased, making it less likely that the starting phase will be measured in error, at the cost of two data bits per symbol.