Parallel Wire Trnasmission Lines
Characteristic Impedance Formulas

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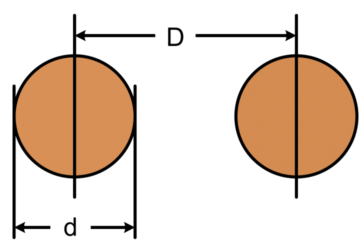
Where it matters, all units are taken to be MKS in this article. Conductors are assumed to have infinite conductivity. *Natural* logarithms are always indicated by the ln() function. Logarithms in any other base use the $\log_b()$ function, and will always have a base, b, specified. Impedances discussed here concern only the lowest order TEM mode of propagation. For values above 400-500 ohms or so, additional modes of propagation become possible, so caveat emptor when dealing with high impedance designs.

Round Conductors



The conductor diameter is represented by the symbol d, and the center-to-center spacing by D. The ratio D/d is unitless, and these dimensions may be expressed in any convenient units (as long as both are expressed in the same unit).

There is a well known, closed form solution to Maxwell's equations for this case. The characteristic impedance when the conductors are embedded in an infinite medium with permeability μ and permittivity ϵ is

$$Z_o = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon}}\cosh^{-1}\frac{D}{d}$$

In air, where relative permittivities and permeabilities are approximately unity,

$$Z_o = \frac{1}{\pi}\sqrt{\frac{\mu_o}{\epsilon_o}}\cosh^{-1}\frac{D}{d} \approx 119.92\cosh^{-1}\frac{D}{d}$$

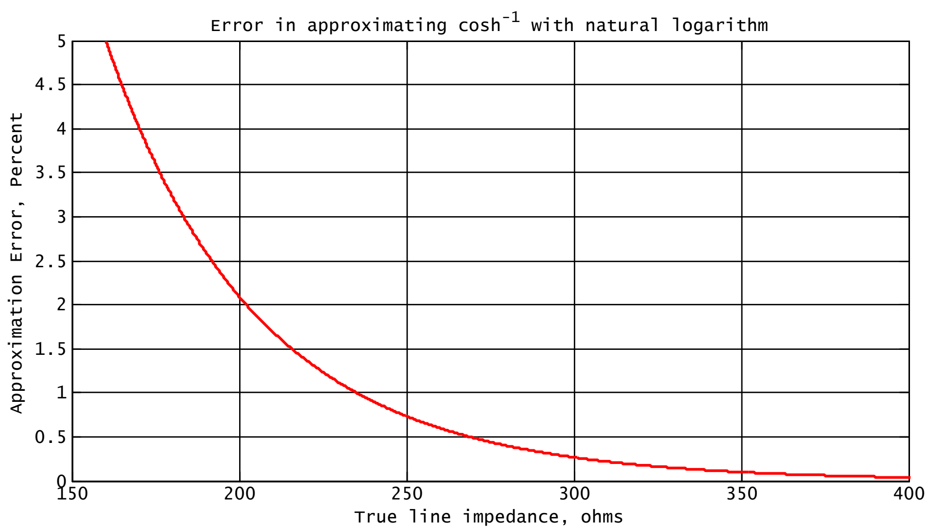
The value of the square root term is the impedance of free space (376.73...Ω), so the multiplier is that impedance divided by π.

A Common Approximation

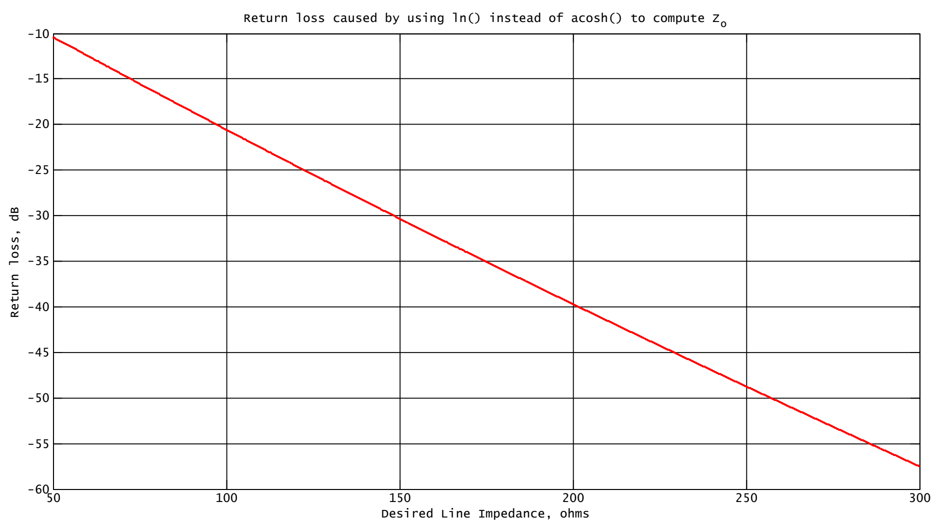
It is not uncommon to find formulas where the hyperbolic arc-cosine has been replaced with the natural logarithm.

$$\cosh^{-1}\frac{D}{d} \approx \ln\frac{D}{d/2}$$

This change works okay for larger values of D/d, but for narrow conductor spacing significant errors are introduced. For unknown reasons, substitution of ln() for $\cosh^{-1}()$ seems to be particularly common in amateur radio circles.

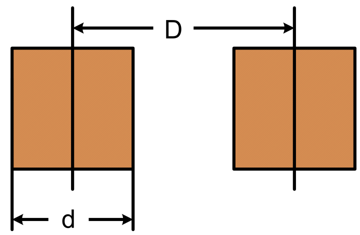


The graph above shows how much error is introduced in the line impedance by using this approximation. The next graph below shows this in terms of return loss due to the impedance error.



For line impedances of 200 ohms or more, the error introduced will result in a return loss no worse than 40dB, and for most purposes this is entirely acceptable. For a line impedance of 100 ohms, the error results in a return loss of about 20dB; this is enough start being a concern in some situations.

Square Conductors



The impedance of a square conductor parallel wire transmission line, with conductors of dimension d on a side, with center-to-center spacing of D might have a closed form solution, but we are not aware of one, nor have we tried to work one out.

For this geometry, E-M simulations have been carefully performed over a range of geometries. The simulated impedance is nearly linear in $(\cosh^{-1} D/d)$. Three equations (one linear and two second-order polynomials) have been fit to the data over different ranges of D/d. Simulations covered a total D/d range from 1.05 to 29, which is an approximate impedance range of 16 to 470 ohms.

1. A linear fit for $D/d > 1.15$ which is useful for most purposes.
2. A more accurate quadratic fit for small values of D/d.
3. A more accurate quadratic fit for larger values of D/d.

The three curve fits are:

$$\Phi = \cosh^{-1} \frac{D}{d}$$

$$Z_{o1}(\Phi) = 121.73\Phi - 25.75 \,|_{D/d>1.15} \quad (1)$$

$$Z_{o2}(\Phi) = 39.82\Phi^2 + 70.56\Phi - 10.23 \,\Big|_{1.05 \geq D/d \leq 1.25} \quad (2)$$

$$Z_{o3}(\Phi) = -0.878\Phi^2 + 125.60\Phi - 28.86\Big|_{D/d \geq 1.25} \quad (3)$$

These may be easily solved for Φ given a desired value of Z_o, and the line spacing determined as $D/d = \cosh \Phi$.

For convenience, the solutions for equations (2) and (3) are provided below. Note that the sign on the square root is not the same for both equations.

$$\Phi_2(Z_o) = -0.88 + \sqrt{1.0418 + 0.0251 Z_o}$$

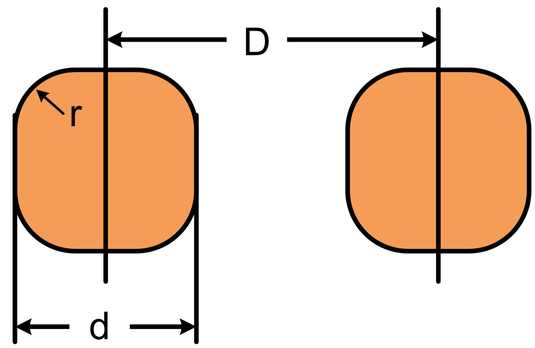
$$\Phi_3(Z_o) = 71.52 - \sqrt{5083.5862 - 1.1390 Z_o}$$

The performance of equations (1-3) in predicting the simulated values is shown in the table below.

Equation	Z_o range	RMS Error	Max Error
1	40 and up	0.8%	1.7%
2	16 to 58	0.3%	0.7%
3	58 and up	0.16%	0.4%

It should be safe to extrapolate equations (1) and (3) values of $D/d > 29$. Taking (2) below D/d values of 1.05 may not be accurate. However, this represents a 16Ω line which is not really practical anyway.

Square Conductors with Rounded Corners



Rounding the corners of square conductors with a smooth radius, r, increases Z_o by the roughly the same number of ohms, regardless of the actual line impedance. Therefore, it has more relative effect on low impedance geometries.

Rounded corners can be characterized by their radius normalized to the conductor width, r/d. A rounding radius equal to half the line width results in a perfectly round conductor. Based on simulations of a line with D/d=1.6 (roughly 100 ohms), the following curve fit estimates how much impedance is added by rounding the corners, as a function of the normalized corner radius.

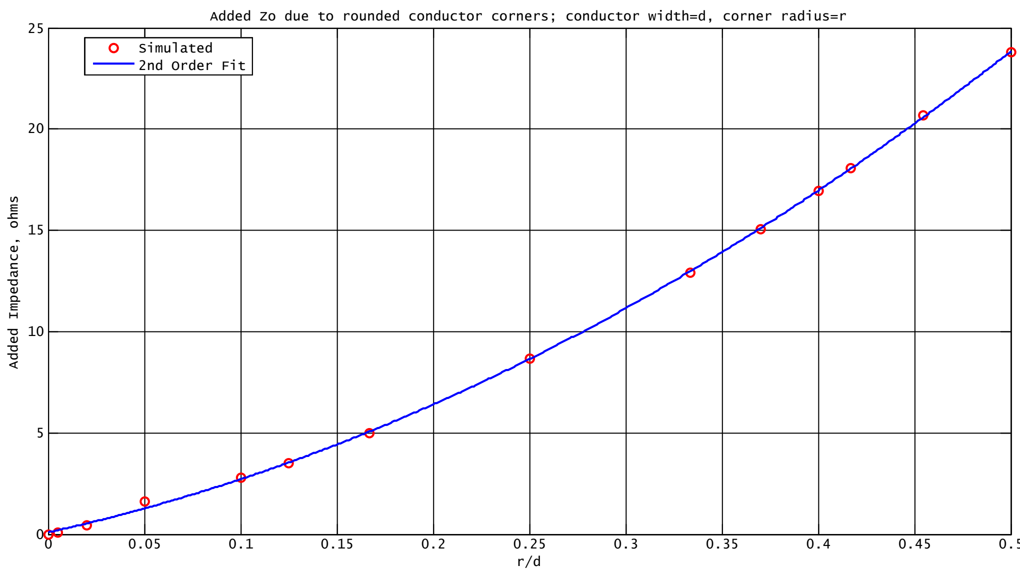
$$\Delta Z_o\left(\frac{r}{d}\right) = 53.06\left(\frac{r}{d}\right)^2 + 20.97\,\frac{r}{d} + 0.09\,\Bigg|_{0\leq r/d\leq 0.5}$$

For this geometry ($D/d = 1.6$), here's how the approximation compares to simulated and theoretical values.

Method	Z_o (Square)	Z_o (Round)
Simulation	101.58	125.37
Std Formula		125.55
BFSL	101.69	
Polynomial	101.68	

Added Impedance	
Simulated	Predicted from polynomial
23.97	23.83

This graph shows the simulated data (red dots) and second order polynomial curve fit.



In summary, to estimate the impedance of a square conductor line with rounded corners, first use one of the curve fits Z_{o1}, Z_{o2}, Z_{o3} to obtain the impedance of the line made with square conductors, then add the impedance predicted by this curve fit:

$$Z_o(D,d,r) = \Delta Z_o\left(\frac{r}{d}\right) + Z_{on}\left(\cosh^{-1}\frac{D}{d}\right)\bigg|_{n=1,2,3}$$

To find the line spacing for a desired impedance using square conductors with known values of r, d, first subtract the rounding correction from the desired impedance, then use one of the solutions to the square conductor curve fits.

$$D = d\ \cosh\left[\Phi_n\left(Z_o - \Delta Z_o\left(\frac{r}{d}\right)\right)\right]_{n=1,2,3}$$

E-M Simulation Specifics

Simulations were performed using perfect electrical conductors and waveguide ports at each end of the transmission line. The ports were sized to be 60 times the largest dimension of the TL profile. Using larger port areas did not seem to have much effect on the calculated port impedances.

Ports were divided into two concentric squares, with a finer mesh applied to the inner square to improve the accuracy of port impedance calculations.

The solver setup specified that residuals should be kept below -60dB, and simulations were run at 50MHz with lines having $d = 0.1$ inch.

No real-world lines were fabricated for testing to verify the simulations of square lines, although simulations of round lines agreed well with theory.

Simulation Check

As a validation of E-M simulations performed for this study, several transmission lines with round conductors were simulated, covering a range of impedances from 38 to 350 ohms. Since the "right answer" is known in this case, it provides a good check on the simulation setup and accuracy. A BFSL fit to the simulation data yielded this formula for the characteristic impedance, which is very close to theory.

$$Z_o = 119.98 \cosh^{-1}\left(\frac{D}{d}\right) - 0.11$$

This formula fits the simulated impedances with an RMS error of 0.16% and a maximum error of 0.6%.